Natural Deduction with Gentzen Sequents

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- Also, as we see with the rule for Implication Introduction, our current ND setup is lacking somewhat in clarity.

- Ideally, we'd like a representation that is more compact and more directly reflects the fact that a certain conclusion is "reachable" or "provable" or "deducible" given a certain set of premises.

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- ND with sequents has an explicit symbol that means "is provable/deducible from" or "follows from" or "entails". This symbol is called the **turnstile** because of its glyph: $\vdash$.
- Recall that an argument is a set of propositions, some of which are premises and at least one of which is the conclusion.
- By adding the turnstile, we're able to compactly write an argument (as we formally define it) on a single line in something called a **sequent**, which has the general form

$$P_0, \ldots, P_n \vdash C \tag{1}$$

where P_0 through P_n are the premises of the argument and C is its conclusion.

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- Now that we have sequents, we can re-write all of our ND rules in this new, more perspicuous format.

Hyp in Sequent Style I

Inference Rule 1 (Hypothesis)

$$\varphi \vdash \varphi$$

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- Notice that Hypothesis just says "φ is provable from itself (alone)".
- So this way of writing the rule already says more clearly that a hypothesis in a proof is a proof of a proposition φ based on no other assumptions other than φ itself.

Hyp in Sequent Style II

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- But notice that you can always tell a sequent is a premise by looking to see if the left side and right side are exactly the same (and you didn't create that sequent by applying any inference rules).
- Now we'll give the rules for the connectives, with introduction rules first and then elimination rules.

→I in Sequent Style I

Inference Rule 2 (Implication Introduction)

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \, (\to \text{I})$$

- In prose, Rule 2 simply says that if you've got a proof of ψ and one of its premises is φ, then you've got a proof of $\varphi \to \psi$ from all the premises you had before *except* φ.

→I in Sequent Style Example

- For example, let's assume you have a proof of D from the premises A, B, and C. Then you can use Rule 2 as follows to get a proof of $B \to D$:

$$\frac{A, B, C \vdash D}{A, C \vdash B \to D} \, (\to\text{I})$$

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- So this rule for Implication Introduction is already a lot simpler than the one we previously had.

→E in Sequent Style

Inference Rule 3 (Implication Elimination)

$$\frac{\Gamma \vdash \varphi \rightarrow \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi} \; (\rightarrow\text{E})$$

- Rule 3 is just the same rule for Implication Elimination that we had before, but now we're keeping track of the premises.

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- In Rule 3, notice that the premises accompanying the implication (Γ) are combined with the premises accompanying the antecedent (Δ) in the resulting conclusion sequent. The notation Γ, Δ means "the premises in Γ combined with the premises in Δ".

→E in Sequent Style Example

- For example, if you have a proof of $B \to D$ from A and C, and also a proof of B from E, then you could use Rule 3 to conclude D:

$$\frac{A, C \vdash B \to D \qquad E \vdash B}{A, C, E \vdash D} \; (\to\text{E})$$

(Note that the left side of the conclusion is just the left side of both of the premises combined together.)

$\wedge$I in Sequent Style

Inference Rule 4 (Conjunction Introduction)

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \wedge \psi} \, (\wedge\text{I})$$

- As for Rule 3, Rule 4 is just our old rule for Conjunction Introduction. The only thing new is that the premises (Γ and Δ) are now kept track of.

∧E in Sequent Style

Inference Rule 5 (Conjunction Elimination 1)

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \; (\wedge E_1)$$

Inference Rule 6 (Conjunction Elimination 2)

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \; (\wedge E_2)$$

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- Rules 5 and 6 are essentially unchanged except for the tracking of the premises.
- They say "if you have a proof of $\varphi \wedge \psi$ from a set of premises Γ, then you have a proof of either φ or ψ by itself from those same premises."

¬¬E in Sequent Style I

- Next there's a new version of our old rule for eliminating double negations. First, though, we'll add a new rule that *introduces* double negations.

Inference Rule 7 (Double Negation Introduction)

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg\neg\varphi} \,(\neg\neg I)$$

Inference Rule 8 (Double Negation Elimination)

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \,(\neg\neg E)$$

¬¬E in Sequent Style II

- Rules 7 and 8 are literally mirror images of each other, and together just express the fact that any proposition φ is equivalent to its double negation $\neg\neg\varphi$.

Some Necessary Bookkeeping I

- Finally, we'll discuss a few **structural** (not inference) rules that essentially just say how the set of premises in an argument is managed.

Structural Rule 1 (Contraction)

$$\frac{\Gamma, \varphi, \varphi, \Delta \vdash \psi}{\Gamma, \varphi, \Delta \vdash \psi}$$

Structural Rule 2 (Permutation)

$$\frac{\Gamma, \varphi, \psi, \Delta \vdash \sigma}{\Gamma, \psi, \varphi, \Delta \vdash \sigma}$$

Some Necessary Bookkeeping II

Structural Rule 3 (Weakening)

$$\frac{\Gamma, \varphi, \Delta \vdash \psi}{\Gamma, \varphi, \sigma, \Delta \vdash \psi}$$

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- Structural Rule 2 simply says that the order of premises in the left side of a sequent doesn't matter–you can move them around however you want, they're all still premises.
- Structural Rule 3, which we probably will never need, just states the fact that you can still prove a conclusion from the same premises if you add in some irrelevant stuff.

Excercises

Problem 1

Assume the following propositions:

- $A \wedge B$
- $B \rightarrow C$
- $((A \wedge B) \rightarrow C) \rightarrow \neg\neg D$
- $D \rightarrow E$

Try to come up with a sequent-style ND proof tree with E as its conclusion. (**Hint:** you'll need (at least) the rules of Implication Introduction and Elimination, Conjunction Elimination, and Double Negation Elimination.)