Natural Deduction with Gentzen Sequents

Scott Martin

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Trouble in Paradise

Overview of our ND system

- So far we have some introduction and elimination rules that let us build, using natural deduction (ND), something the represents an argument.
- But, for one thing, it still requires some effort to see the arguments in these representations because you have to know that the premises are at the very top, the conclusion at the very bottom, and everything in the middle is an inference step.
- Also, as we see with the rule for Implication Introduction, our current ND setup is lacking somewhat in clarity.
- Ideally, we'd like a representation that is more compact and more directly reflects the fact that a certain conclusion is "reachable" or "provable" or "deducible" given a certain set of premises.

Sequent-style Natural Deduction

Introducing Sequents I

- To fix some of these problems, we'll adopt an ND in the (Gentzen)[1] sequent style.
- ND with sequents has an explicit symbol that means "is provable/deducible from" or "follows from" or "entails". This symbol is called the **turnstile** because of its glyph: $\vdash$.
- Recall that an argument is a set of propositions, some of which are premises and at least one of which is the conclusion.
- By adding the turnstile, we're able to compactly write an argument (as we formally define it) on a single line in something called a **sequent**, which has the general form

  $$P_0, \ldots, P_n \vdash C \tag{1}$$

  where P_0 through P_n are the premises of the argument and C is its conclusion.

[1] So called after its creator, Gerhard Gentzen.

Introducing Sequents II

- We read a sequent like the one in Equation 1 as "C is provable from the assumptions P_0 through P_n" or "P_0 through P_n entail C".
- A note on meta-language: we'll often write sequents as something like

$$\Gamma \vdash \varphi$$

  where φ is a meta-variable over propositions (as before), and Γ is taken to represent a (possibly empty) set of propositions.
- Now that we have sequents, we can re-write all of our ND rules in this new, more perspicuous format.

Hyp in Sequent Style I

Inference Rule 1 (Hypothesis)**.**

$$\varphi \vdash \varphi$$

- Rule 1 is the familiar way to introduce assumptions into a proof.
- Notice that Hypothesis just says "φ is provable from itself (alone)".
- So this way of writing the rule already says more clearly that a hypothesis in a proof is a proof of a proposition φ based on no other assumptions other than φ itself.

Hyp in Sequent Style II

- Given that Hypothesis is the only rule where the premises are the same as the conclusion, combined with the fact that hypotheses always occur at the very top of a proof, we'll stop writing the (Hyp) label above our premises.
- But notice that you can always tell a sequent is a premise by looking to see if the left side and right side are exactly the same (and you didn't create that sequent by applying any inference rules).
- Now we'll give the rules for the connectives, with introduction rules first and then elimination rules.

→I in Sequent Style I

Inference Rule 2 (Implication Introduction)**.**

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \ (\to\text{I})$$

- In prose, Rule 2 simply says that if you've got a proof of ψ and one of its premises is φ, then you've got a proof of $\varphi \to \psi$ from all the premises you had before *except* φ.

→I in Sequent Style Example

- For example, let's assume you have a proof of D from the premises A, B, and C. Then you can use Rule 2 as follows to get a proof of $B \to D$:

$$\frac{A, B, C \vdash D}{A, C \vdash B \to D} \ (\to\text{I})$$

  (Notice that it doesn't matter which premise you "pull out" of the left side.)
- So this rule for Implication Introduction is already a lot simpler than the one we previously had.

$\to$E in Sequent Style

Inference Rule 3 (Implication Elimination)**.**

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \psi}\ (\to\text{E})$$

- Rule 3 is just the same rule for Implication Elimination that we had before, but now we're keeping track of the premises.
- In Rule 3, notice that the premises accompanying the implication (Γ) are combined with the premises accompanying the antecedent (Δ) in the resulting conclusion sequent. The notation Γ, Δ means "the premises in Γ combined with the premises in Δ".

$\to$E in Sequent Style Example

- For example, if you have a proof of $B \to D$ from A and C, and also a proof of B from E, then you could use Rule 3 to conclude D:

$$\frac{A, C \vdash B \to D \qquad E \vdash B}{A, C, E \vdash D}\ (\to\text{E})$$

(Note that the left side of the conclusion is just the left side of both of the premises combined together.)

$\wedge$I in Sequent Style

Inference Rule 4 (Conjunction Introduction)**.**

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \wedge \psi}\ (\wedge\text{I})$$

- As for Rule 3, Rule 4 is just our old rule for Conjunction Introduction. The only thing new is that the premises (Γ and Δ) are now kept track of.

$\wedge$E in Sequent Style

Inference Rule 5 (Conjunction Elimination 1)**.**

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ (\wedge\text{E}_1)$$

Inference Rule 6 (Conjunction Elimination 2)**.**

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ (\wedge\text{E}_2)$$

- Rules 5 and 6 are essentially unchanged except for the tracking of the premises.
- They say "if you have a proof of $\varphi \wedge \psi$ from a set of premises Γ, then you have a proof of either φ or ψ by itself from those same premises."

¬¬E in Sequent Style

- Next there's a new version of our old rule for eliminating double negations. First, though, we'll add a new rule that *introduces* double negations.

Inference Rule 7 (Double Negation Introduction)**.**

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg\neg\varphi}\ (\neg\neg\text{I})$$

Inference Rule 8 (Double Negation Elimination)**.**

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ (\neg\neg\text{E})$$

- Rules 7 and 8 are literally mirror images of each other, and together just express the fact that any proposition φ is equivalent to its double negation $\neg\neg\varphi$.

Structural Rules

Some Necessary Bookkeeping

- Finally, we'll discuss a few **structural** (not inference) rules that essentially just say how the set of premises in an argument is managed.

Structural Rule 1 (Contraction)**.**

$$\frac{\Gamma, \varphi, \varphi, \Delta \vdash \psi}{\Gamma, \varphi, \Delta \vdash \psi}$$

Structural Rule 2 (Permutation)**.**

$$\frac{\Gamma, \varphi, \psi, \Delta \vdash \sigma}{\Gamma, \psi, \varphi, \Delta \vdash \sigma}$$

Structural Rule 3 (Weakening)**.**

$$\frac{\Gamma, \varphi, \Delta \vdash \psi}{\Gamma, \varphi, \sigma, \Delta \vdash \psi}$$

Structural Rules Explained

- These rules may look confusing, but mostly they're just a formality.
- Notice, for one thing, that we don't even bother to write labels for these rules. That's mainly because we're rarely going to explicitly invoke these rules.
- Structural Rule 1 just says that a premise occurring more than once doesn't matter–it's either a premise of the argument or it isn't. In practice, we'll just make sure not to repeat premises.
- Structural Rule 2 simply says that the order of premises in the left side of a sequent doesn't matter–you can move them around however you want, they're all still premises.
- Structural Rule 3, which we probably will never need, just states the fact that you can still prove a conclusion from the same premises if you add in some irrelevant stuff.

Homework

Excercises

Problem 1. Assume the following propositions:

- $A \wedge B$
- $B \to C$
- $((A \wedge B) \to C) \to \neg\neg D$
- $D \to E$

Try to come up with a sequent-style ND proof tree with E as its conclusion. (**Hint:** you'll need (at least) the rules of Implication Introduction and Elimination, Conjunction Elimination, and Double Negation Elimination.)