Using Natural Deduction to Represent Arguments (Part 1)

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February 13, 2012

Propositions, Arguments, and Logic

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  b. If Clarence does catch a Walleye, Myrtle will cook it for dinner.
  c. Myrtle didn't go shop for dinner.
  d. So, Myrtle is cooking Walleye for dinner.

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- Notice that this meets our definition of an argument because
  1. All the sentences in (1) are declaratives, and
  2. There are some premises (namely, (1a-1c)) and a conclusion (1d).

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- With these three atomic propositions, we can represent (1) as follows:

$$\neg W \rightarrow S \tag{1a}$$

$$W \rightarrow C \tag{1b}$$

$$\neg S \tag{1c}$$

$$C \tag{1d}$$

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- This is OK, but ideally we don't want to rely on writing things out in prose all of the time.
- How can we make this process of deduction more formal and precise?

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- Such a system should let us distinguish arguments with true premises (sound arguments) from ones that don't have true premises, and arguments whose conclusions follow from their premises (valid arguments) from ones whose conclusions don't follow from their premises.
- Natural Deduction (ND) is a system for encoding syntactically what we can find out about validity using truth tables. So it's a kind of shorthand.

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- ND uses **inference rules** to express validity so we don't have to keep referencing the truth tables for the connectives.
- These deductive rules give us a way to combine the premises of the argument in (1) to see that its conclusion is, in fact, valid.

The Rules

- In all of our ND rules, φ and ψ represent propositions in PL. These rules follow the general format

$$\frac{P_0 \qquad \dots \qquad P_n}{C}\,(L)$$

where P_0 through P_n are premises, C is the conclusion, and L is a label corresponding to the name of the inference rule that was invoked.

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- Rules are repeatedly applied to make larger arguments from smaller ones.

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- This rule is so simple it almost seems trivial.
- But it is in some ways the most powerful rule we have–it's the way a premise is introduced in any argument.

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- (which is OK, because remember that in general we're not interested in the truth or falsehood of an argument's premises, just what can be deduced from them.)
- For the informal argument in (1), we would need to use Hypothesis to introduce the complex propositions denoted by (1a) through (1c).

Implication Elimination

- Rule 2, sometimes called Modus Ponens, just says that if you've proved (or assumed) an implication and you've also proved the antecedent of that implication, then you've proved the consequent.

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- Given rule 2, we can already represent part of the process of deduction used in the informal argument in (1).

$$\dfrac{\dfrac{}{\neg S \to \neg\neg W}\ (\mathrm{Hyp}) \qquad \dfrac{}{\neg S}\ (\mathrm{Hyp})}{\neg\neg W}\ (\to\mathrm{E})$$

Figure 1: Partial proof of the argument in (1).

Implication Elimination (cont.)

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- Notice that the last inference step of the above proof is just an instance of Rule 2 with $\neg S$ inserted for φ and $\neg\neg W$ inserted for ψ.
- The first inference steps of this proof are both instances of Rule 1 (Hypothesis).

Using PL Equivalences

- Sometimes we can rewrite PL sentences as other PL sentences they are equivalent to.
- We did this in our informal reasoning process to turn $\neg W \to S$ into $\neg S \to \neg\neg W$ so that we could get things to work out right.

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- We'd ideally like to be able to write proofs so that deduction steps always invoke rules, not equivalences.
- That is, we'd like to write equivalences only in the assumptions and let the inference rules do all of the deductive "work" for us.

Double Negation Elimination

- So to turn $\neg\neg W$ into W, we'll use a special rule:

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- Rule 3 expresses in syntactic terms what we could easily verify using truth tables–that the truth value of any PL sentence φ is the same as its double negation $\neg\neg\varphi$.

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Figure 2: Full proof of the argument in (1).

- Notice that the proof in Figure 1 is a subproof of this proof (this should be easy to see from its shape and the propositional letters/rule labels it contains).

Exercises I

Consider the following informal argument:

(2) a. If a man owns a Lake Wobegon car dealership, he is the brother of the man who owns the other Lake Wobegon car dealership.

b. Clint owns Lake Wobegon's Ford dealership.

c. Clarence owns Lake Wobegon's Chevy dealership.

d. Clint and Clarence are brothers.

And also, a simplified version of it:

(3) a. If a man owns a Lake Wobegon car dealership, he is the brother of the man who owns the other Lake Wobegon car dealership.

b. Clint owns a Lake Wobegon car dealership.

Exercises II

c. Clint's brother owns Lake Wobegon's other car dealership.

Problem 1

Answer the following questions about the argument in (2):

a. Write down all the propositions expressed in the argument.
b. Which propositions expressed by this argument are premises, and which are the conclusion?
c. If you needed to diagram the pattern of reasoning reflected in this argument, which inference rules from our ND rules would probably be used?

Problem 2

Translate the informal argument in (3) into a formal proof using the rules introduced in Rules 1–3.