Entailments and Equivalence

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- But we'd also like to be able to discover equivalences between propositions expressed in propositional logic (PL).
- Also, we'd like to have a way to say (formally) when some premise(s) entail some conclusion.

Proving What We've Known All Along

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- Notice that you *could* use truth tables to quickly convince yourself that $\varphi \wedge \psi$ is true in all the same cases that $\psi \wedge \varphi$ is true.
- But with ND, we don't need to do everything by interpretation anymore. We can give a **syntactic** proof (shown in Figure 1) that says the same thing without needing to consider every way the world could be.

$$\dfrac{\dfrac{\dfrac{}{A \wedge B}\,(\text{Hyp})}{B}\,(\wedge\text{E}_2) \qquad \dfrac{\dfrac{}{A \wedge B}\,(\text{Hyp})}{A}\,(\wedge\text{E}_1)}{B \wedge A}\,(\wedge\text{I})$$

Figure 1: Proof of $B \wedge A$ from $A \wedge B$.

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- Notice how this mirrors the notion of entailment for deductively valid arguments–starting with true premises, you arrive at a conclusion that must be true (no matter how the world is).
- We can mention this in the meta-language, but sometimes we'd actually like to use the fact that some premise(s) entail some conclusion in yet another proof.
- One way to do this is to simply re-use the conclusion as a new premise (we'll do this a lot).

Introducing Implication

- Another way is to make an implication using one of the premises as the antecedent and the conclusion as the consequent (Rule 7 formalizes this step).

Inference Rule 7 (Implication Introduction)

$$\dfrac{\dfrac{}{[\varphi]_i}\,(\mathrm{Hyp}) \quad \vdots \quad \psi}{\varphi \to \psi}\,(\to\mathrm{I})\ i$$

Rule 7 Explained

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- For bookkeeping, we pick an as-yet-unused number i (it doesn't matter which) to label the introduction step. We then write $[\quad]_i$ around the withdrawn hypothesis, and the same number i to the left of the introduction step.
- Notice that our rules are getting a bit sloppy–now we have to know that $\vdots$ means something like "any number of inference steps".
- So the rule of Implication Introduction is not as formally rigorous as some of the other rules we've used up to this point because it relies more on the meta-language.

Example of Rule 7 In Action

- Figure 2 shows an example of this rule in action.

$$
2\;\dfrac{1\;\dfrac{\dfrac{\dfrac{}{[A \to B]_2}\,(\text{Hyp}) \qquad \dfrac{\dfrac{}{[A \wedge C]_1}\,(\text{Hyp})}{A}\,(\wedge\text{E}_1)}{B}\,(\to\text{E})}{(A \wedge C) \to B}\,(\to\text{I})}{(A \to B) \to ((A \wedge C) \to B)}\,(\to\text{I})
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- Notice that the choice to start numbering at 1 gives us an easy mnemonic to remember the order the introductions occurred in.
- Also note that the $[\quad]_i$ makes it clear that the withdrawn premises were only assumed "for the time being" and are no longer required assumptions for the truth of the final conclusion.

Exercises

Problem 1

Give a formal ND proof of $\neg\neg A \to B$ that assumes as premises only $\neg\neg A$ and $A \to B$.