Functions

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Functions

- A relation F between A and B is called a **(total) function** from A to B iff for every $x \in A$, there exists a unique $y \in B$ such that $x \, F \, y$.
- In that case we write $F \colon A \to B$.
- This is often expressed by saying that F **takes** members of A as **arguments** and **returns** members of B as **values** (or, alternatively, **takes its values** in B).
- Clearly, $\mathsf{dom}(F) = A$.
- For each $a \in \mathsf{dom}(F)$, the unique b such that $a \, F \, b$ is called the **value** of F **at** a, written $F(a)$.
- Alternatively, we say F **maps** a to b, written $F \colon a \mapsto b$.
- It is easy to prove that there is a unique set, called B^A, whose members are the functions from A to B.

Basic Definitions for Functions

Suppose $F \colon A \to B$, $A' \subseteq A$, and $B' \subseteq B$. Then

- the **restriction** of F to A' is the function from A' to B given by

$$F \restriction A' = \{\langle u, v \rangle \in F \mid u \in A'\}$$

- The **image** of A' by F is the set

$$F[A'] =_{\text{def}} \{y \in B \mid \exists x \in A'(y = F(x))\}$$

- The **preimage** (or **inverse image**) of B' by F is the set

$$F^{-1}[B'] =_{\text{def}} \{x \in A \mid \exists y \in B'(y = F(x))\}$$

This is more simply described as
$\{x \in A \mid F(x) \in B'\}$

Operations

For any $n \geq 0$, an n-ary **(total) operation** on A is a function from $A^{(n)}$ to A.

- Many of the useful operations we will encounter are binary operations, i.e. functions from $A \times A$ to A.
- A unary operation on A is just a function from A to A.
- A nullary operation on A is a function from 1 to A.

Identity Functions

For any A, the identity relation id_A is a unary operation on A, such that, for any $x \in A$,

$$\mathsf{id}_A(x) = x$$

Function Composition

- If $F\colon A \to B$ and $G\colon B \to C$, then their (relational) composition $G \circ F$ is a function from A to C, namely

$$G \circ F = \{\langle x, z \rangle \in A \times C \mid \exists y \in B(y = F(x) \wedge z = G(y))\}.$$

- For each $x \in A$,

$$G \circ F(x) = G(F(x))$$

Here n is any natural number. The set $2 = \wp(1) = \{0, 1\}$ is often called the set of **truth values**.

- There is a unique function from A to 1, called $\square_A$.
- There is a unique function from 0 to A, called $\lozenge_A$.
- The **successor** function **suc** is the unary operation on ω that maps each natural number to its successor.
- **Arithmetic** functions such as **addition** ($+$), **multiplication** ($\cdot$), and **exponentiation** ($\star$), are binary operations on ω.

  Soon we'll show how these are defined *recursively*, but first we will need to introduce the Recursion Theorem (RT).

More Workhorse Functions (2/2)

- For each function $F \colon A \to 2$, the **kernel** of F is the subset
$$\ker(f) =_{\text{def}} \{x \in A \mid f(x) = 1\}$$
and for each $B \in \wp(A)$, the **characteristic function of B in A** is the function that maps each $x \in A$ to 1 if $x \in B$, and to 0 if $x \in A \setminus B$.

- The members of A^n are called A-**strings of length** n. These are indispensible for formalizing theories of phonology and syntax.

- Operations on 2 are called **truth functions**. These are used to define the meanings of the FOL logical connectives such as $\neg$, $\wedge$, $\vee$, and $\to$. and in defining the references of linguistic expressions.

- For any set A, we can define on $\wp(A)$ the unary operation of **complement**, and the binary operations of **union**, **intersection**, and **relative complement** (exercise).

Kinds of Functions

Suppose $F \colon A \to B$. Then F is called:

- **injective**, or **one-to-one**, or an **injection**, if it maps distinct members of A to distinct members of B
- **surjective**, or **onto**, or a **surjection**, if $\mathsf{ran}(F) = B$
- **bijective**, or **one-to-one and onto**, or a **bijection**, or a **one-to-one correspondence**, if it is both injective and surjective.

  Note: For any bijection $F \colon A \to B$, we can show that the inverse relation F^{-1} is also a function, and in fact a bijection, from B to A.
- A relation F between A and B is called a **partial function** from A to B provided there is a subset $A' \subseteq A$ such that F is a (total) function from A' to B.

Examples of Injective Functions

- for $A \subseteq B$, the function $\mu_{A,B} \colon A \to B$ that maps each member of A to itself, called the **embedding** of A into B

- the functions ι_1 and ι_2, called **canonical injections**, from the cofactors A and B of a cartesian coproduct $A + B$ into the coproduct, defined by $\iota_1(a) = \langle 0, a \rangle$ and $\iota_2(b) = \langle 1, b \rangle$ for all $a \in A$ and $b \in B$

Examples of Surjective Functions

- The **projections** π_1 and π_2 of a cartesian product $A \times B$ onto its factors A and B respectively, defined by $\pi_1(\langle a, b \rangle) = a$ and $\pi_2(\langle a, b \rangle) = b$ for all $a \in A$ and $b \in B$.
- Given a set A with an equivalence relation $\equiv$, the function from A to $A/\equiv$ that maps each member of A to its equivalence class

Examples of Bijective Functions

- any identity function
- We can prove that **suc** is a bijection from ω to the set $\omega \setminus \{0\}$ of positive natural numbers.
- For any A, there is a bijection from $\wp(A)$ to 2^A that maps each subset of A to its characteristic function.

  The inverse of this bijection maps each characteristic function to its kernel.
- The truth function that maps 0 and 1 to each other
- The complement operation on a powerset
- For any set A, there is a bijection from A to the set A^1 that maps each $a \in A$ to the nullary operation that maps 0 to a.
- More generally, for any $n \in \omega$, there is a bijection from $A^{(n)}$ to A^n that maps each A-string of length n to an n-tuple of elements of A.

Propositional Functions

- In linguistic semantics, operations on the set P of propositions are used to define the senses of 'logic words' such as *and*, *implies*, and *it is not the case that*.

- More generally, 'genuine' relations (such as loving, owning, being at, and knowing that), as opposed to mathematical relations, are modelled as functions whose values are propositions.

Modelling Word Senses with Propositional Functions

- Recall that in our foundations for linguistic semantics, we have assumed that we have a set P of **propositions**, a set W of **worlds**, and a (mathematical) relation @ between propositions and worlds.

- We now assume additionally that we have a set I of **individuals**.

- We then model the sense of 'relational' expressions (such as verbs, predicate adjectives, common nouns, and determiners) by functions from a cartesian product $A_1 \times \ldots \times A_n$ to P, where $n > 0$ is the number of arguments and the choices of the A_i depend on what kinds of things (individuals, propositions, functions from individuals to propositions, etc.) are being related.