Interpreting Propositional Logic (Part 1)

Scott Martin

August 2, 2010

Giving an Interpretation for a Sentence of PL

- Analyzing arguments requires that we first have a systematic way to discover the (in)validity of sentences of PL.
- At a bare minimum, we want our interpretation of PL to be decisive: interpreting a PL sentence should be unambiguous.
- A couple of problems with doing this:
  1. The validity of a complex PL sentence is always dependent on the validity of its component atomic PL sentences. But we can't always know whether all the atomic sentences are true or false!
  2. The syntax of PL is recursive, so a PL sentence can be arbitrarily large. Given any two PL sentences S and T, we can always form $\neg S$, $S \wedge T$, $T \rightarrow S$, etc.

Truth Assignments: Ways Things Could Be

- To handle problem 1 above, we'll need to consider every possible way things could be.
- That is, given that we can't always know the truth value of each atomic proposition, we need to devise a scheme for discover what the truth value of a complex proposition *would be* just in case we *did* know what the truth values of all its component atomic propositions were.
- To that end, we look at the simplest case: a single atomic proposition (call it A). Since A is a proposition, it must have a truth value, and so we know there are only two ways things could be (call them w_1 and w_2):

	A
w_1	T
w_2	F

  Here, the **truth assignments** w_1 and w_2 capture all the possible truth values for A: either A is true (w_1) or else it is false (w_2).
- The next most complicated case is a situation with two atomic propositions A and B. Now we have to consider four separate cases:

	A	B
w_1	T	T
w_2	T	F
w_3	F	T
w_4	F	F

In this case, both A and B could be true (or false) and A could be true with B false or vice versa.

- This is an instance of a general pattern: each time we consider another atomic proposition, the number of ways things could be doubles. That is, for a sentence of PL containing n atomic propositions, there are 2^n ways things could be.

- To take a more concrete example:

(1) a. Pastor Ingqvist likes lutefisk.
b. Evelyn likes Powdermilk Biscuits.
c. Florian likes Walleye.

Let L be the proposition expressed by (1a), P the proposition expressed by (1b), and W the proposition expressed by (1c). Then there are $2^3 = 8$ possible truth assignments:

	L	P	W
w_1	T	T	T
w_2	T	T	F
w_3	T	F	T
w_4	T	F	F
w_5	F	T	T
w_6	F	T	F
w_7	F	F	T
w_8	F	F	F

Suppose we happen to know that Evelyn does indeed like Powdermilk Biscuits and Florian really likes Walleye but that Pastor Ingqvist actually can't stand lutefisk. Then the truth assignment w_5 corresponds to how things are in the real world. But more generally, we'd like to know what *would have* happened in the other cases.

Truth Tables: (In)validity of Arbitrarily Complex PL Sentences

- Notice that no matter how complex a sentence of PL is, we still interpret it as either true or false.

- That is, no matter how we build up a complex PL sentence, it is still just a proposition.

- We also know that although there are infinitely many possible complex propositions, there are only finitely many *ways* of connecting atomic propositions to form complex ones (namely, five: $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$).

- So dealing with problem 2 above just means saying what each of the connectives does to the truth values of the proposition(s) (atomic or complex) it is connecting.
- Interpreting a complex proposition depends on three things:
  1. The atomic propositions it contains,
  2. The connectives used to put them together, and
  3. The way they are combined (i.e., $(A \to B) \to C$ and $A \to (B \to C)$ are different sentences and should get different interpretations).

Negation ($\neg$)

- Negating a proposition toggles (reverses) its truth value. (Since negation operates on a single proposition, it is called a **unary connective**.)
- That is, if a proposition P is true (false), then $\neg P$ is false (true).
- We capture this fact in the **truth table** for negation (shown in Table 1). This truth table

φ	$(\neg\varphi)$
T	F
F	T

Table 1: Truth table for negation.

  says that for a given (atomic or complex) PL sentence φ, every truth assignment that assigns T for φ also assigns F for $\neg\varphi$ and vice versa.
- Negation in PL is used to represent the English usages of negation found in *not*, *it is not the case that*, etc.
- To see if this interpretation of negation corresponds with our intuitions about how language and reasoning interact, consider

  (2) Clint sees Myrtle.

  Let M be the proposition expressed by (2). Then without knowing *whether* M is true or not, we know that *if* M is true then *Clint does not see Myrtle* (i.e., $\neg M$) is false. Likewise, if M is false, then $\neg M$ must be true.

Homework

Problem 1. Given an argument that depends on four distinct atomic propositions, how many possible truth assignments are there for those atomic propositions?

Problem 2. Assume that a certain argument is based on only four atomic propositions: A, B, C and D. Write out all the possible truth assignments that argument could have.

Problem 3. Let S be a sentence of PL. To know the truth value of $(\neg S)$, do we have to know what the truth value of S is? Why or why not?