# Proof of the Existence of Cartesian Products

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**Proposition.** *For any two sets $A$ and $B$, there exists a set, which we call the Cartesian product of $A$ and $B$ (written $A \times B$), whose members are all $\langle a, b \rangle$ such that $a \in A$ and $b \in B$.*

*Proof.* Given two sets $A$ and $B$, we first use Separation to construct a set containing all and only sets of the form $\{\{a\}, \{a, b\}\}$ where $a$ is a member of $A$ and $b$ is a member of $B$. We then use Extensionality to ensure that there is only one such set.

Let $C$ be $A \cup B$. By Powerset, among the members of $\wp(C)$ are $\{a\}$ where $a \in A$ and $\{a, b\}$ where $a \in A$ and $b \in B$. Then for every $a \in A$ and $b \in B$, $\wp(\wp(C))$ contains (at least) all sets of the form $\{\{a\}, \{a, b\}\}$. We then use Separation to prove the existence of a set (call it P) that contains *only* sets of that form:

$$\mathrm{P} = \{\{\{a\}, \{a, b\}\} \in \wp(\wp(A \cup B)) \mid a \in A \text{ and } b \in B\}$$

By the definition of ordered pair, the members of P are all the ordered pairs $\langle a, b \rangle$ where $a \in A$ and $b \in B$. There is only one such set P by Extensionality; this set is the Cartesian product of $A$ and $B$. $\square$